9. Completions

9.1 Inverse Limits

Def: A directed set is a nonempty partially ordered set $(I, \leq)$ s.t. $\forall i,j \in I: \exists k \in I: i \leq k$ and $j \leq k$

Exm: $(\mathbb{N}_0, \leq)$, more generally, any totally ordered set
$(\mathbb{N}, |)$

Def: I directed set. An inverse system over I is a family of ab. groups (modules, rings) $(A_i)_{i \in I}$ together with, for every pair $j \leq i$ in I, a hom. $f_{ij}: A_i \to A_j$ s.t. $\forall i: f_{i,i} = id_{A_i}$ and the hom's are compatible, i.e., $\forall i,j,k: j \leq i, k \leq j \Rightarrow f_{ik} = f_{jk} \circ f_{ij}$

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ik}"] \arrow[dr, "f_{ij}"'] & & A_k \\
& A_j \arrow[ur, "f_{jk}"'] &
\end{tikzcd}$$

Exm: 1) For $I = (\mathbb{N}_0, \leq)$:
i) K field, $A_i := K[x]/(x^i)$, $A_j \to A_i$ for $j > i$ canonical epi
ii) $\mathbb{Z}/p^j\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^i\mathbb{Z}$ for $j > i$, $(A_i = \mathbb{Z}/p^i\mathbb{Z})$

2) $I = (\mathbb{N}, |)$: $A_i := \mathbb{Z}/i\mathbb{Z}$, if $i \mid j$, there is a canonical epi $\mathbb{Z}/j\mathbb{Z} \to \mathbb{Z}/i\mathbb{Z}$

Def: Let $(A_i)_{i \in I}$, $(f_{ij})_{j \leq i}$ be an inverse system. An ab. group (module, ring) A together with hom. $g_i: A \to A_i$

An ab.group (module, ring) A together with hom. $g_i : A \to A_i$ is the inverse limit of the system if

1) The system $(g_i)_{i\in I}$ is compatible, i.e., $f_{ij} \circ g_i = g_j \quad \forall j \leq i$

$$\begin{tikzcd} & A \arrow[dl, "g_i"'] \arrow[dr, "g_j"] & \\ A_i \arrow[rr, "f_{ij}"'] & & A_j \end{tikzcd}$$

[UP]

2) If B is another ab.group (module, ring) w. a compatible system of hom's $h_i : B \to A_i$ (as in (1)), there exists a unique hom $h : B \to A$ s.t. $h_i = g_i \circ h \quad \forall i \in I$

$$\begin{tikzcd} & & A_j \\ B \arrow[r, "h"] \arrow[urr, bend left, "h_j"] \arrow[drr, bend right, "h_i"'] & A \arrow[ur, "g_j"] \arrow[dr, "g_i"'] & \\ & & A_i \arrow[uu, "f_{ij}"'] \end{tikzcd}$$

Notation: $A = \varprojlim A_i$

Remark: Inverse limits are unique up to unique isomorphism by the UP.

Exm $\varprojlim K[x]/(x^i) = K[[x]]$

Prop 9.1 If $(A_i)_{i\in I}$, $(f_{ij})_{j\leq i}$ is an inverse system of abelian groups $\varprojlim A_i$ exists. If the A_i are rings or modules over a fixed ring R, $\varprojlim A_i$ is a ring or R-module.

Proof Sketch: $\varprojlim A_i := \{ (a_i)_{i\in I} \in \prod_{i\in I} A_i : \forall j \leq i : f_{ij}(a_i) = a_j \}$

with componentwise operation.

$\Longrightarrow g_i : \varprojlim A_i \to A_i, \ (a_i)_{i\in I} \mapsto a_i$ is a hom.,

$\Rightarrow g_i : \varprojlim A_i \to A_i,\ (a_i)_{i\in I} \mapsto a_i$ is a hom.,

- $f_{ij} \circ g_i = g_j$ by definition if $j \le i$
- If B is a group [ring, module] w. compatible system $h_i : B \to A_i$, then $h : B \to \varprojlim A_i,\ b \mapsto (h_i(b))_{i\in I}$ is a well-defined hom, unique with $g_i \circ h = h_i$.

An additional ring/module structure carries over. $\square$

Exm: 1) $\mathbb{Z}_p := \varprojlim\limits_{i\in\mathbb{N}_0} \mathbb{Z}/p^i\mathbb{Z}$ is the (ring) of p-adic integers

2) If $I = (\mathbb{N}, |)$

$\hat{\mathbb{Z}} := \varprojlim\limits_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is the ring of profinite integers

Exc: $\hat{\mathbb{Z}} \cong \prod\limits_{p\in\mathbb{P}} \mathbb{Z}_p$. (using CRT: $\mathbb{Z}/n\mathbb{Z} \cong \prod\limits_{p\in\mathbb{P}} \mathbb{Z}/p^{v_p(n)}\mathbb{Z}$)

Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ be inverse systems, and $\varphi_i : A_i \to B_i$ hom's compatible w. the system maps: if $j \le i$,

$$\begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "\varphi_i"'] & A_j \arrow[d, "\varphi_j"] \\ B_i \arrow[r, "g_{ij}"'] & B_j \end{tikzcd}$$

commutes.

Then we get a hom $\varphi : \varprojlim A_i \to \varprojlim B_i$, s.t.

$$\begin{tikzcd} \varprojlim A_i \arrow[r, "\varphi"] \arrow[d, "g_i"'] & \varprojlim B_i \arrow[d, "h_i"] \\ A_i \arrow[r, "\varphi_i"'] & B_i \end{tikzcd}$$

commutes (use UP)

This respects composition & identity hom's, i.e., $\varprojlim$ is a functor

This respects composition & identity hom's, i.e., $\varprojlim$ is a functor from inverse systems over I to groups [rings, modules].

<u>Thm 9.2</u>: Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$, $(C_i)_{i\in I}$ be inverse systems of abelian groups.[*] Suppose there are exact sequences $0 \to A_i \xrightarrow{\varphi_i} B_i \xrightarrow{\psi_i} C_i$ compatible with the system maps (i.e., a left exact sequence of inverse systems).

(1) This induces an exact sequence $0 \to \varprojlim A_i \xrightarrow{\varphi} \varprojlim B_i \xrightarrow{\psi} \varprojlim C_i$

(2) If $I = (\mathbb{N}_0, \leq)$, each ψ_i is surjective, and the maps f_{ij} for $(A_i)_{i\in I}$ are surjective, then ψ is surjective.

[*] Or more generally, R-modules over some ring R.

<u>Proof</u>: (1)

$$\begin{array}{ccc} (a_i)_{i\in I} & \longmapsto & (\varphi_i(a_i))_{i\in I} \\ \prod_{i\in I} A_i & \longrightarrow & \prod_{i\in I} B_i \\ \cup\!\mid & & \cup\!\mid \\ \varprojlim A_i & \xrightarrow{\varphi} & \varprojlim B_i \end{array}$$

Since each φ_i is injective, so is φ.

Similarly $\psi \circ \varphi = 0$, since $\psi_i \circ \varphi_i = 0 \ \forall i \in I$.

<u>$\ker(\psi) \subseteq \mathrm{im}(\varphi)$</u>: Let f_{ij}, $j \leq i$, denote the system maps of $(A_i)_{i\in I}$, g_{ij} of $(B_i)_{i\in I}$, h_{ij} of $(C_i)_{i\in I}$.

Let $(b_i)_{i\in I} \in \varprojlim B_i$, i.e., $g_{ij}(b_i) = b_j \ \forall j \leq i$.

Suppose $\psi((b_i)_{i\in I}) = 0 \Rightarrow \psi_i(b_i) = 0 \ \forall i$

$\Rightarrow b_i = \varphi_i(a_i)$ with $a_i \in A_i$ [the a_i's are unique by injectivity of φ_i]

<u>Show</u>: $f_{ij}(a_i) = a_j \ \forall j \leq i$, then $(a_i)_{i\in I} \in \varprojlim A_i$

$$f_{ij}(a_i) = a_j \iff \underbrace{\varphi_j(f_{ij}(a_i))}_{\substack{= g_{ij}(\varphi_i(a_i)) \\ = g_{ij}(b_i)}} = \underbrace{\varphi_j(a_j)}_{= b_j}$$

$g_{ij}(b_i)$

and $g_{ij}(b_i) = b_j$ because $(b_\ell)_{\ell\in I}$ is a compatible system.

(2) Let $(c_i)_{i\in I} \in \varprojlim C_i$, i.e., $h_{ij}(c_i) = c_j$ if $j\leq i$.

<u>Goal:</u> Construct $(b_i)_{i\in I}$ s.t. $\psi_i(b_i) = c_i$ and $g_{ij}(b_i) = b_j$ if $j\leq i$

Since $I = \mathbb{N}_0$, only need $\psi_i(b_i) = c_i$ and $g_{i,i-1}(b_i) = b_{i-1}$ if $i\geq 1$

<u>Recursive construction</u> <u>$i=0$:</u> ψ_0 surj., so let $b_0 \in B_0$ s.t. $\psi_0(b_0) = c_0$

<u>$i\geq 1$, $i-1\to i$:</u> Let $\tilde{b}_i \in B_i$ s.t. $\psi_i(\tilde{b}_i) = c_i$.

For $x \in \ker(\psi_i)$, $\psi_i(\tilde{b}_i + x) = c_i$ and

$$g_{i,i-1}(\tilde{b}_i + x) = g_{i,i-1}(\tilde{b}_i) + g_{i,i-1}(x) \quad (\overset{!}{=} b_{i-1})$$

$$\psi_{i-1}(g_{i,i-1}(\tilde{b}_i) - b_{i-1}) = h_{i,i-1}(\underbrace{\psi_i(\tilde{b}_i)}_{=c_i}) - \underbrace{\psi_{i-1}(b_{i-1})}_{=c_{i-1}} = 0,$$

so $g_{i,i-1}(\tilde{b}_i) - b_{i-1} \in \ker(\psi_{i-1}) = \mathrm{im}(\varphi_{i-1})$.

$\Rightarrow g_{i,i-1}(\tilde{b}_i) - b_{i-1} = \varphi_{i-1}(a_{i-1})$ with $a_{i-1} \in A_{i-1}$

$\overset{f_{i,i-1}\text{ surj}}{\Longrightarrow} a_{i-1} = f_{i,i-1}(a_i)$ with $a_i \in A$

$\Rightarrow \varphi_{i-1}(a_{i-1}) = \varphi_{i-1}(f_{i,i-1}(a_i)) = g_{i,i-1}(\varphi_i(a_i))$.

$\Rightarrow b_{i-1} = g_{i,i-1}(\tilde{b}_i - \varphi_i(a_i))$, so $x := -\varphi_i(a_i)$, i.e. $b_i := \tilde{b}_i - \varphi_i(a_i)$

works. $\square$

9.2 Topological Groups

<u>Def:</u> An abelian topological group is an abelian group G, s.t. G is also a top. space and $G\times G \to G$, $(a,b)\mapsto a+b$ and $G\to G$, $a\mapsto -a$ are continuous.

↖product topology

<u>Exm:</u> • $(\mathbb{R},+)$ w. Euclidean topology

• $S^1 = \{z\in\mathbb{C} : |z| = 1\}$ as multiplicative group $(\cong \mathbb{R}/\mathbb{Z})$

• Any abelian group w. discrete topology

- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ as multiplicative group $(\cong \mathbb{R}/\mathbb{Z})$
- Any abelian group w. discrete topology

<u>Remark</u>: $\forall a \in G$: $\quad G \to G,\ x \mapsto a + x$ is a homeomorphism, so carries nbhds of 0 bij. onto nbhds of a. The topology is determined by nbhds of 0.

<u>Lemma 9.3</u> G ab. top. group, $H := \bigcap \{U : U \text{ nbhd of } 0\}$

(1) $H \leqslant G$

(2) $H = \overline{\{0\}}$

(3) G Hausdorff $\Leftrightarrow$ $\{0\}$ closed.

(4) G/H is Hausdorff.

<u>Proof</u>: (1) $0 \in H$ ✓ Let $a, b \in H$. Let U be an open nbhd of 0,

$\alpha : G \times G \to G,\ (x,y) \mapsto x - y \overset{\alpha \text{ cont.}}{\Longrightarrow} \alpha^{-1}(U)$ is open and $(0,0) \in \alpha^{-1}(U)$.

$\Rightarrow \alpha^{-1}(U)$ contains a set of the form $V_1 \times V_2 \ni (0,0)$, V_i open.

$a, b \in V_i \Rightarrow a - b = \alpha(a,b) \in U$, <u>so</u>: $H - H \subseteq H$.

(2) V is a nbhd of x $\Leftrightarrow$ $V = x \pm U$, U nbhd of 0

So: $x \in H \Leftrightarrow x \in U = 0 + U \quad \forall$ nbhds U of 0

$\Leftrightarrow 0 \in x - U \quad \forall$ nbhds U of 0

$\Leftrightarrow x \in \overline{\{0\}}$

(3) „$\Rightarrow$" In Hausdorff spaces points are closed. Explicitly:

If $x \neq 0$, there exists an open nbhd U_x of x s.t. $0 \notin U$

$\Rightarrow \{0\} = \bigcap_{x \in G \setminus \{0\}} G \setminus U_x$

„$\Leftarrow$" A top. space X is Hausdorff $\Leftrightarrow$ $\Delta = \{(x,x) : x \in X\} \subseteq X \times X$ is closed.

Let $\Delta = \{(a,a) : a \in G\}$, $\quad \alpha : G \times G \to G,\ (x,y) \mapsto x$

is closed.

Let $\Delta = \{(g,g) : g \in G\}$, $\alpha : G \times G \longrightarrow G$, $(x,y) \mapsto x-y$

$\Rightarrow \Delta = \alpha^{-1}(\{0\}) \underset{\{0\}\text{ closed}}{\overset{\alpha\text{ cont.}}{\Longrightarrow}} \Delta$ closed.

(4) G/H is a top. group w.r.t. quotient topology. The points are cosets $a+H$ and hence closed by (2).

$\overset{(3)}{\Longrightarrow} G/H$ Hausdorff. $\square$